Loop Quantum Gravity and Spin Foam Models

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Why QUANTUM GENERAL RELATIVITY.

The world is QUANTUM and GENERAL RELATIVISTIC.

It is not just quantum and not just general relativistic

Quantum field theories (STANDARD MODEL) and general relativity are in perfect agreement with all known observations.

Einstein equations can not be consistently coupled to quantum fields.

There is NO physical justification for looking for new interactions.

It works in 2-dimensions: Liouville gravity Dynamical triangulations and in 3-dimensions: Witten's solution of Chern-Simons Theory Ponzano-Regge model and/or Turaev-Viro invariant of 3-manifolds

The problem, as seen by a high energy physicist.

The U(1)xSU(2)xSU(3) standard model perfectly describe virtually everything we can measure - except gravity (the smallest of all the interactions).

Use the same strategy for gravity. The metric field is just another fields over e fixed background

The problem, as seen by a relativist.

The key lesson of GR is that there is no background metrics over which physics happens. Gravitational field=space-time Background independent formulations of quantum field theory. No poincarre‘ group representation theory. The concept of elementary particle is missing.

A quote form Roger Penrose, The theory of quantized directions:

A reformulation is suggested in which quantities normally requiring continuous coordinates for their descriptions are eliminated from primary consideration. In particular, since space and time have therefore to be eliminated, what might be called a form of machs principle be invoked: only relationships of objects to each other can have significance.

Assumption of loop quantum gravity:

- Quantum mechanics and general relativity
- Backgroud independence
- No unification
- Four spacetime dimension and no supersymmetry

These assumptions are shared with other approaches to quantum gravity !!!

The difference is in the choice of the algebra of function that we want to promote to quantum operator

LOOP QUANTUM GRAVITY $\Rightarrow$ ALGEBRA OF LOOPS

Loop quantum gravity is not the only attempt along this line:
Dynamical triangulations
Regge calculus
Simplicial models
...

Loop quantum gravity is a serious proposal for the construction of a:

BACKGROUND INDEPENDENT, CONSISTENT, THEORY OF QUANTUM EINSTEIN EQUATIONS.

- It assumes that general relativity and quantum mechanics are booth correct descriptions of our world.
- It gives an intriguing pictures of plank scale space time.

Main unsolved problem:

- Dynamics (which super-Hamiltonian constraints)
- Scaling properties
- Continum Limit

QUANTUM GRAVITY. C. Rovelli. 2004, Cambridge, UK: Univ. Pr. (2004) 455 p.

BACKGROUND INDEPENDENT QUANTUM GRAVITY: A STATUS REPORT.
A. Ashtekar, J. Lewandowski, Class.Quant.Grav.21:R53,2004

R. De Pietri, Canonical "loop" quantum gravity and spin foam models, in "Recent Developments in General Relativity", B. Casciaro, D. Fortunato, M. Francaviglia eds.(Springer-Verlag, Milano, 2000). xxx-archive:gr-qc/9903076.

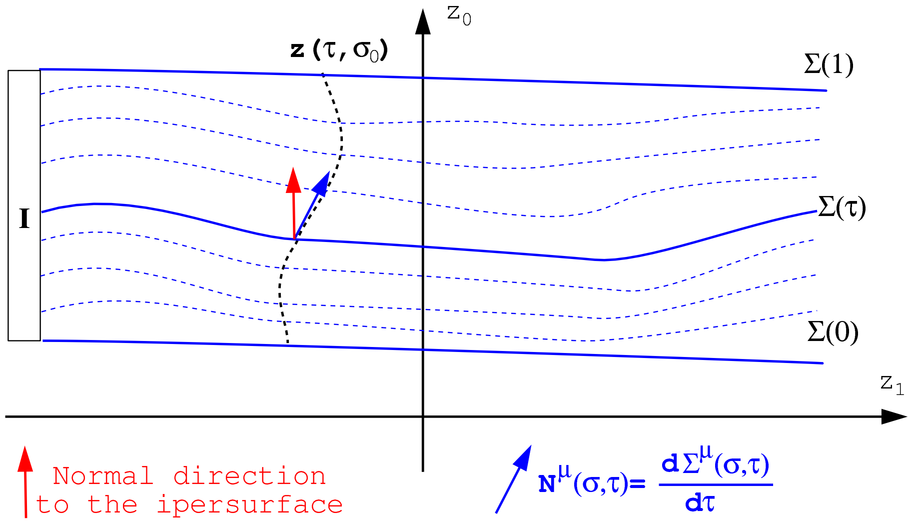
Canonical Quantum Gravity

In General Relativity there is no time !!

Way out: introduce a foliation of the space time manifold; use the parameter labeling the leaf of the foliation as time

CONFIGURATION VARIABLE: the restriction of the space-time metrics to the leaves

Canonical VARIABLE: the space metrics + estrinsic curvature of the leaf



DYNAMICS: 4 constraints: 3 space-diffomorphism 1 time-diffeomorphism

The strategy is analogous the one followed by Bjorken-Drell for quantizing the electromagnetic field: Consider the Hamiltonian formulations of the Electromagnetic Field The system has a constraint (The Gauss Law) Quantize the system. Impose the constraints.

Use New variables (connection formulation)

Hamiltonian formulation of classical general relativity in terms of the canonically conjugated variables $\tilde{E}_i^a(x)$ (densitized triad fields) and $A_a^i(x)$ (Ashtekar's connection)

$$C_i(x) = \mathcal{D}_a \tilde{E}_i^a(x) \simeq 0 \qquad \text{(Gauge constraints)}$$

$$\mathcal{H}_a(x) = \tilde{E}_i^b(x) F_{ab}^i(x) \simeq 0 \qquad \text{(Diffeomorphism constraints)}$$

$$\mathcal{H}(x) = \frac{\epsilon^{ijk}}{\sqrt{g}} \tilde{E}_i^a(x) \tilde{E}_j^b(x) F_{ab}^k(x) \simeq 0 \qquad \text{(Hamiltonian constraints).}$$

Fundamental variables to be promoted to WELL DEFINED Quantum Operators

$$\mathcal{T}_\gamma[A_a^i(x)] = \mathrm{Tr}\left[\exp\left(\int_\gamma A_a(x) dx^a\right)\right] \tag{1}$$

Determination of the **physical** Hilbert space $\mathcal{H}_{\text{Phys}}$, involves the analysis of the following hierarchy of Hilbert spaces:

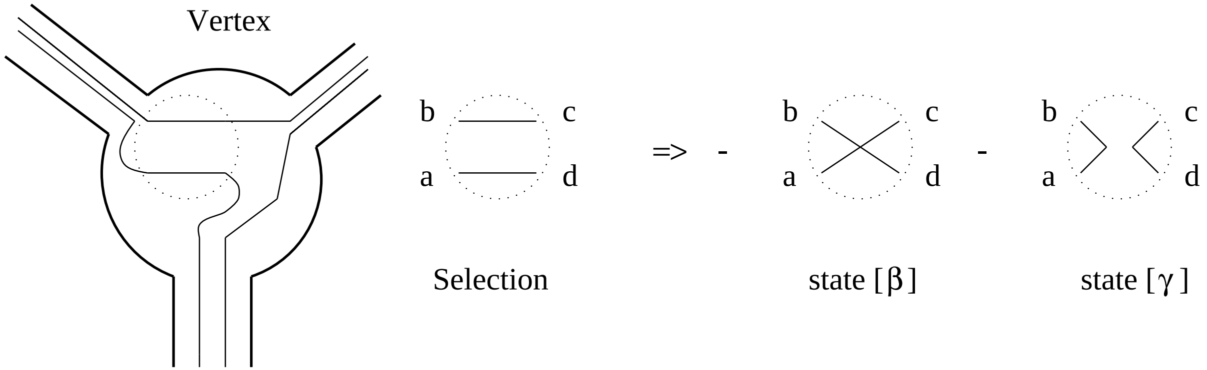
$$\mathcal{H} \overset{C_i(x)}{\Longrightarrow} \mathcal{H}_{\text{aux}} \overset{\mathcal{H}_a(x)}{\Longrightarrow} \mathcal{H}_{\text{Diff}} \overset{\mathcal{H}(x)}{\Longrightarrow} \mathcal{H}_{\text{Phys}}$$

where an arrow means imposition of a constraint

The space $\mathcal{H}_{\text{aux}}$

Algebraic construction of the vector space. Use the same strategy of the Harmonic oscillator creation operator.

Non linear relation between the operator that correspond to the Wilson's loops: MANDELSTAM RELATIONS ⇒ SPIN-NETWORKS.

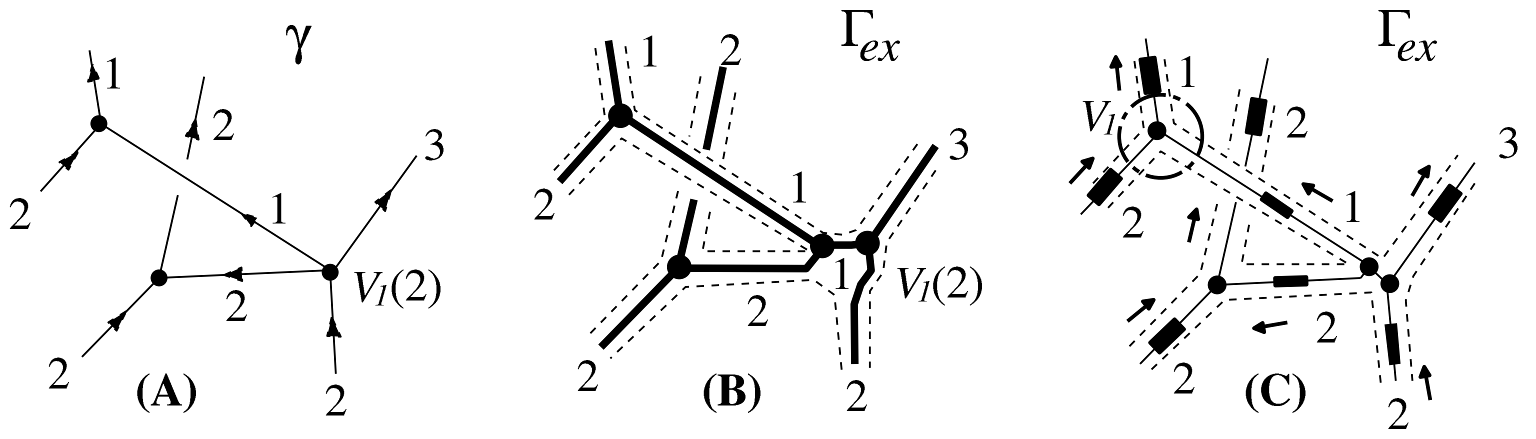


Solve the SU(2) Mandelstam constraints:

Use graphical TANGLE theoretic techniques [Rovelli Smolin, Phys.Rev.D52(1995)5743, De Pietri Rovelli, Phys.Rev. D54(1996)2664, KauffmanLins 1994 Temperley-Lieb recoupling theory and invariant of 3-manifolds]

Use group theoretic techniques [Ashtekar Lewandowsk, Jour. Geom Phys. D54(1995)2664, Baez 96 in a book]

A basis on $\mathcal{H}_{\text{aux}}$ is the spin-network basis



The gauge invariant Hilbert space $\mathcal{H}_{\text{aux}}$ is the projective limit of the gauge invariant sub-sector of the Hilbert spaces $\mathcal{H}_\gamma$. Moreover, the Peter-Weyl theorem gives us a natural basis in this space of gauge invariant functions $f_\gamma(g_{e_1},\ldots,g_{e_n})$, the spin networks basis:

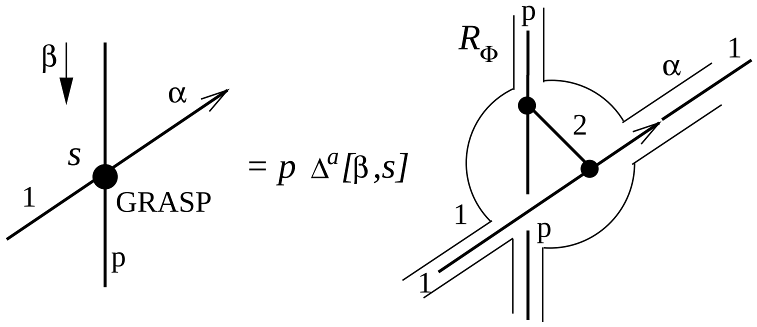
$$f_\gamma(g_{e_1},\ldots,g_{e_n}) = \sum_{\vec{\pi},\vec{\iota}} f(\gamma,\vec{\pi},\vec{\iota}) \quad \mathcal{T}_{\gamma,\vec{\pi},\vec{\iota}}\,[A], \qquad \mathcal{T}_{\gamma,\vec{\pi},\vec{\iota}}[A] \stackrel{def}{=} \left(\otimes_{i=1}^{\#\text{edge}} \pi_i(g_{e_i})\right)\cdot\left(\otimes_{j=1}^{\#\text{vertex}} \iota_j\right) \quad (2)$$

where

$$g_{e_i} = \mathcal{P} e^{\int_{\gamma_i} dx^k A_k^i(x)\hat{\tau}_i}$$

The action of the Triad operator.

Tangle theoretic:



The space $\mathcal{H}_{\text{Diff}}$

SOLVE THE 3-DIFFEMORPHISM CONSTRAINTS $\Rightarrow$ KNOT CLASS

The essential idea is that on $\mathcal{H}_{\text{aux}}$ there is a natural definition of the action of a diffeomorphism $\phi \in \text{Diff}$ ($\phi : M \to M$). In fact, in the spin-networks basis we can associated to each diffeomorphism ϕ the operator

$$\hat{U}_\phi |S\rangle = \hat{U}_\phi |\gamma, \vec{\pi}, \vec{\iota}\,\rangle = |\phi(S)\rangle = |S'\rangle = |\phi(\gamma), \vec{\pi}, \vec{\iota}\,\rangle, \tag{3}$$

where $S' = \phi(S)$ is the imagine under ϕ of the spin network S. At this point one can define the class of spin networks knots s as the equivalence class of spin networks under diffeomorphism, i.e. $S, S' \in s$ it it exists a $\phi \in \text{Diff}$ such that $S' = \phi(S)$. In this way, we can define $\mathcal{H}_{\text{Diff}} = \mathcal{H}_{\text{aux}}/\text{Diff}$ with the scalar product:

$$\langle s|s'\rangle = N \int [D\phi] \langle S|\hat{U}_\phi S'\rangle = N \int [D\phi] \langle \phi(S)|S'\rangle \tag{4}$$

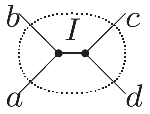
Results depending only on the structure of $\mathcal{H}_{\mathrm{Diff}}$

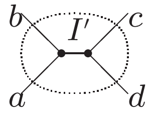
Spectrum of geometric operators

Example: The Volume operator

$$\hat{V}^2_{\mathcal{RS}} = \frac{1}{3!}\sum_{\mathbf{e}_I\cap\mathbf{e}_J\cap\mathbf{e}_K=v}\left|\frac{\mathrm{i}}{16}\hat{W}_{[IJK]}\right| \quad , \qquad \hat{V}^2_{\mathcal{AL}} = \left|\frac{\mathrm{i}}{16}\frac{1}{3!}\sum_{\mathbf{e}_I\cap\mathbf{e}_J\cap\mathbf{e}_K=v}\epsilon(\mathbf{e}_I,\mathbf{e}_J,\mathbf{e}_K)\hat{W}_{[IJK]}\right| \; .$$

where, $\hat{W}_{IJK} = 2\ \epsilon_{ijk}\ X^i_I X^j_J X^k_K$, and $\epsilon(\mathbf{e}_I,\mathbf{e}_J,\mathbf{e}_K) = \mathrm{sgn}(\det(\dot{e}_I(0),\dot{e}_J(0),\dot{e}_K(0)))$, $g_I = j_I(\bar{A}(\mathbf{e}_l))$, and $X_I = X(g_I)$ is the right invariant vector field on $SU(2)$. All edges are chosen as outgoing.

$\hat{W}_{[012]}\Big|$ 

$\Big\rangle = \sum_{I'} W^{(4)}_{[012]}(a,b,c,d)_I{}^{I'}\Big|$ 

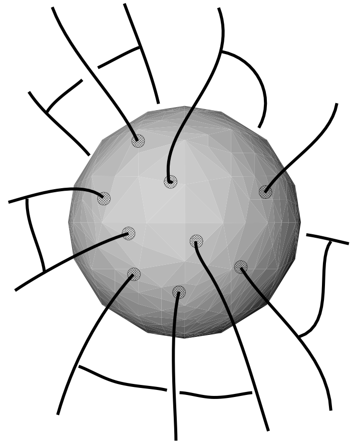
$\Big\rangle$, (5)

and:

$$\tilde{W}^{(4)}_{[012]}(a,b,c,d)^{t+\epsilon}_{t-\epsilon} = -\epsilon(-1)^{\frac{a+b+c+d}{2}}\left[\frac{1}{4t(t+2)}\frac{a+b+t+3}{2}\frac{c+d+t+3}{2}\right.$$
$$\left.\frac{1+a+b-t}{2}\frac{1+a+t-b}{2}\frac{1+b+t-a}{2}\frac{1+c+d-t}{2}\frac{1+c+t-d}{2}\frac{1+d+t-c}{2}\right]^{\frac{1}{2}}$$

Black hole entropy

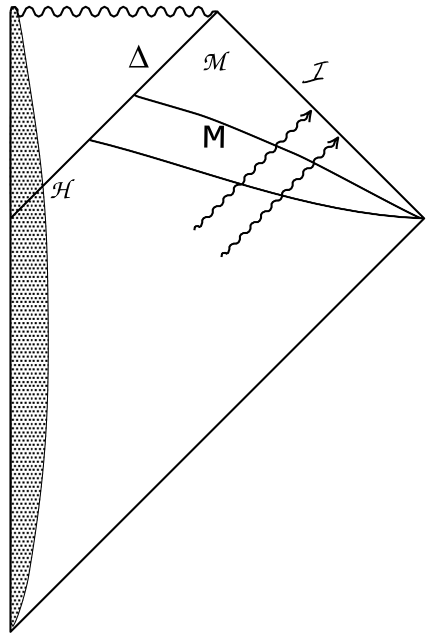
The main idea is to count the number of states that correspond to a surface of area A. Some progres in imposing that the surface is the horizon of a black-hole.



[Rovelli, Phys. Rev. Lett. 14(1996)3288, Krasnov, Phys. Rev. D55(1997)3505, Ashtekar, Baez, Corici, Krasnov, Phys. Rev. Lett. 80(1998)904]

+/- of the derivation:
The result is (just) proportional to the accepted Beckenstin-Hawking black hole formula.

$$\frac{\ln 2}{4\pi\gamma l_{\mathrm{Pl}}^2} \leq \frac{\ln N(A)}{A} \leq \frac{\ln 2}{4\pi\gamma l_{\mathrm{Pl}}^2}$$



BLACK HOLE ENTROPY FROM QUANTUM GEOMETRY.
M. Domagala, J. Lewandowski, Class.Quant.Grav.21:5233-5244,2004

Loop quantum cosmology

Many reviews... I'm not going to speak about this subject.

Super Hamiltonian Constraints

Thiemann '96 was able to construct an operator with the *naivë* classical limit.

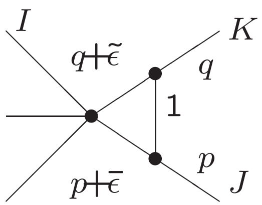
What it is interesting for us it is that:

1. it is possible to define such operator;

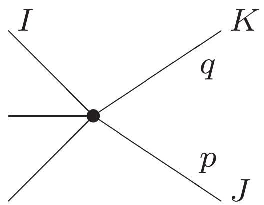
2. a rough idea of what should be its action $\mathcal{H}_{\mathsf{Diff}}$.

Its main property is that it acts locally around each vertex of the spin network. It adds or deletes an edge of color 1

$$\widehat{A}_{vIJK\overline{\epsilon}\tilde{\epsilon}} = \Bigg\langle$$



$$\Bigg| \, \widehat{\mathcal{H}}[N] \, \Bigg|$$



$$\Bigg\rangle \qquad (6)$$

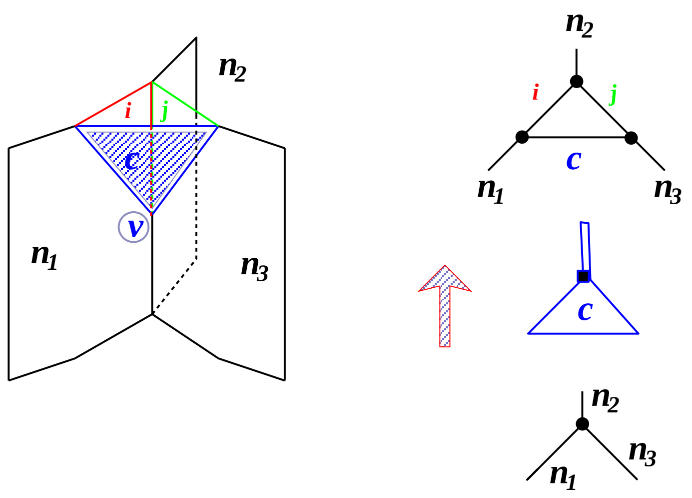
with well definite weight factors [($\widehat{A}_{vIJK\overline{\epsilon}\tilde{\epsilon}}$) matrix elements of the operator] that can be explicitly computed (see Borrisov, De Pietri, Rovelli '97).

PROBLEMS:

1. the physical correctness of the Thiemann's proposal for the regularization of the super-Hamiltonian constraint has been questioned;
2. more of one variant of this operator can be construct and indeed the problem of determine the correct one;
3. there is not a clear understanding of the kernel of the super-Hamiltonian constraint nor of the class of operator that are well defined on this Hilbert space.

However:

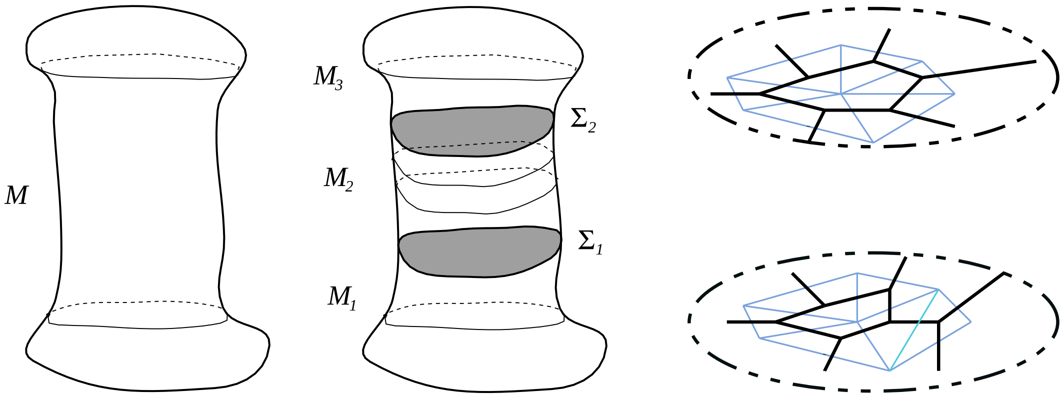
- Nice 3+1 interpretation of it actions

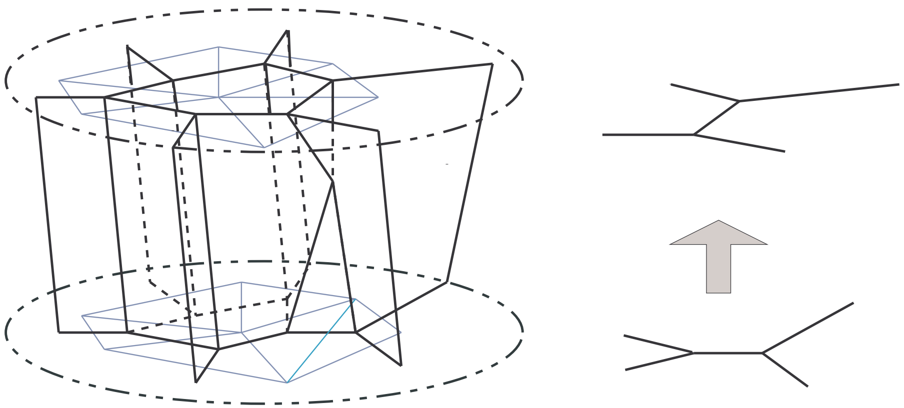


Brings naturaly to the introduction of the concept of evolving Spin-networks, i.e.,

SPIN FOAM MODELS

SPIN FOAM MODELS (General Idea)





- Introduce 2 boundary in a Manifolds
- Let the spin networks to live on the boundaries
- Join the spin networks on the boundaries using a 2-dim branched surface
- The action of the Super-Hamiltonian gives the branching rule

SPIN FOAM MODELS: route (1)

Rovelli Reisemberger Idea:

$$\frac{d}{dt}|S', \Sigma(1)\rangle = \left(N(x,\tau)\widehat{\mathcal{H}}(x) + N^a(x,\tau)\widehat{\mathcal{H}}_a(x)\right)|S, \Sigma(t)\rangle$$

$$|S', \Sigma(1)\rangle = \mathcal{T}\mathrm{exp}\left[-\mathrm{i}\int_0^T d\tau \int d^3x \left(N(x,\tau)\widehat{\mathcal{H}}(x) + N^a(x,\tau)\widehat{\mathcal{H}}_a(x)\right)\right]|S, \Sigma(0)\rangle \quad .$$

The last equation suggests to define the transition amplitude, averaged over all the possible embedding (bordism) interpolating from $\Sigma(0)$ and $\Sigma(1)$:

$$Z[S', S; 1] = \langle S', \Sigma(1)|S, \Sigma(0)\rangle =$$
$$= \int [dN(x,\tau)][dN^i(x,\tau)]\langle s'|\mathcal{T}\mathrm{exp}\left[-\mathrm{i}\int_0^1 d\tau \int d^3x \left(N(x,\tau)\widehat{\mathcal{H}}(x) + N^a(x,\tau)\widehat{\mathcal{H}}_a(x)\right)\right]|S\rangle \quad .$$

The integrations $[dN(x,\tau)][dN^i(x,\tau)]$ are over all the lapse and shift functions that represent a bordism between $\Sigma(0)$ and $\Sigma(1)$.

They note that it is possible to give a well defined meaning to the integration over the lapse of the exponential of infinitesimal diffeomorphism in term of the gauge averaging procedure used to solve the diffeomorphism constraints:

$$\hat{U}_{\phi_{N^i}} =: \exp\left[-\mathrm{i}\int d^3x \; N^a(x)\hat{\mathcal{H}}_a(x)\right] \tag{7}$$

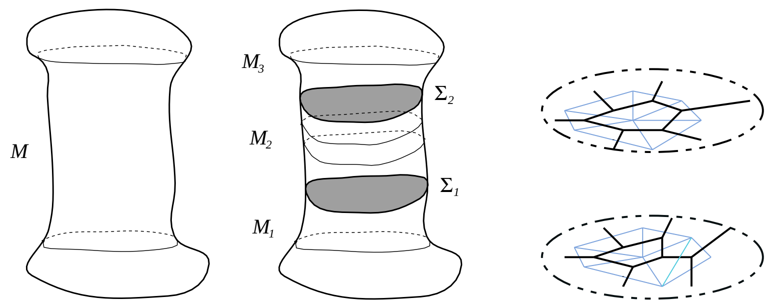
$$|s\rangle := \int [D\phi] \; \hat{U}_\phi |S\rangle =: \int [dN^a(x)] \; \hat{U}_\phi |S\rangle \tag{8}$$

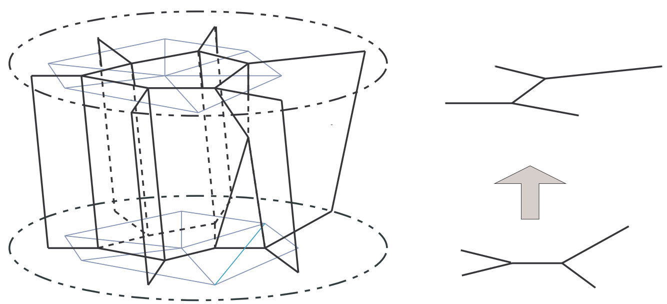
and then, using a series of formal argument, that the definition (7) depends only on the s-knot classes s and s' of S and S', respectively. Morover, they show that it can be expressed in term of a diff-invariant definition (as the one proposed by Thiemann) of the super-Hamiltonian constraint and of its associated proper time propagator:

$$Z_{RR}[s', s; 1] = \int dT < s'|e^{-\mathrm{i} \; T \; \mathcal{H}}|s >= \int dT \sum_\sigma \frac{(\mathrm{i} \; T)^{n(\sigma)}}{n(\sigma)} \prod_{(\text{coloring of } \sigma)} \prod_{v\in[\sigma]} A_v(\sigma) \tag{9}$$

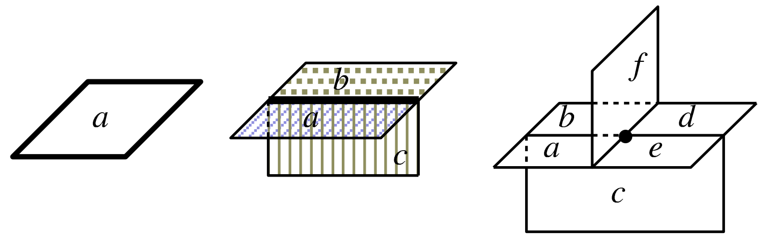
SPIN FOAM MODELS (route 2)

Give the rule for colloring the various objects that appear in the 2 scheleton dual to a decomposition of a Manifold



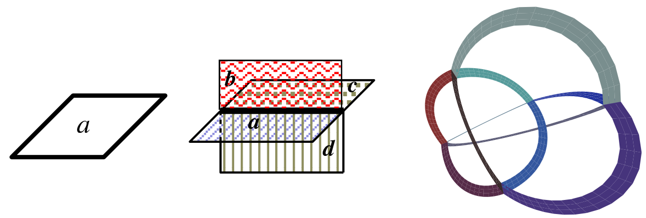


Object to be associated to:



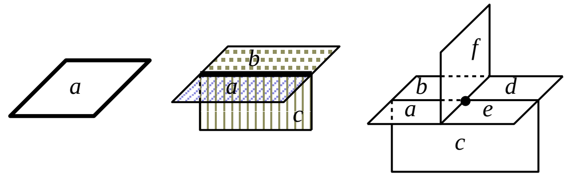
3d: Type **I** (**2dim–object**) Type **II** (**1dim–object**) Type **III** (**0dim–object**)

The three possible configurations of the spine dual to a four dimensional triangulation



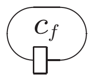
4d: Type **I** (**2dim–object**) Type **II** (**1dim–object**) Type **III** (**0dim–object**)

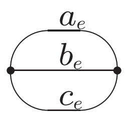
The Turaev-Viro Invariant

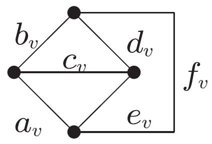


Type **I** (2dim−object) Type **II** (1dim−object) Type **III** (0dim−object)

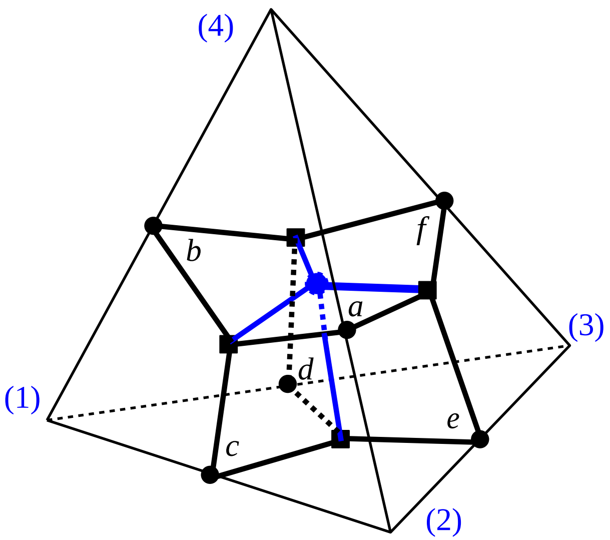
.

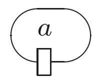
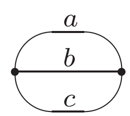
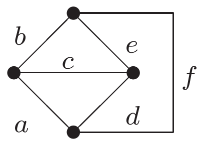
$$TV_G[\mathcal{S}_2] = \Lambda^{-\#(3-cells)} \sum_{\vec{c}\in D(\mathcal{S}_2,G)} \Big[\prod_{f\in\mathcal{F}(\mathcal{S}_2)} \Big($$


$$\Big)^{\chi(f)} \Big] \Big[\prod_{e\in\mathcal{E}(\mathcal{S}_2)} \Big($$


$$\Big)^{-\chi(e)} \Big] \Big[\prod_{v\in\mathcal{V}(\mathcal{S}_2)}$$


$$\Big]$$

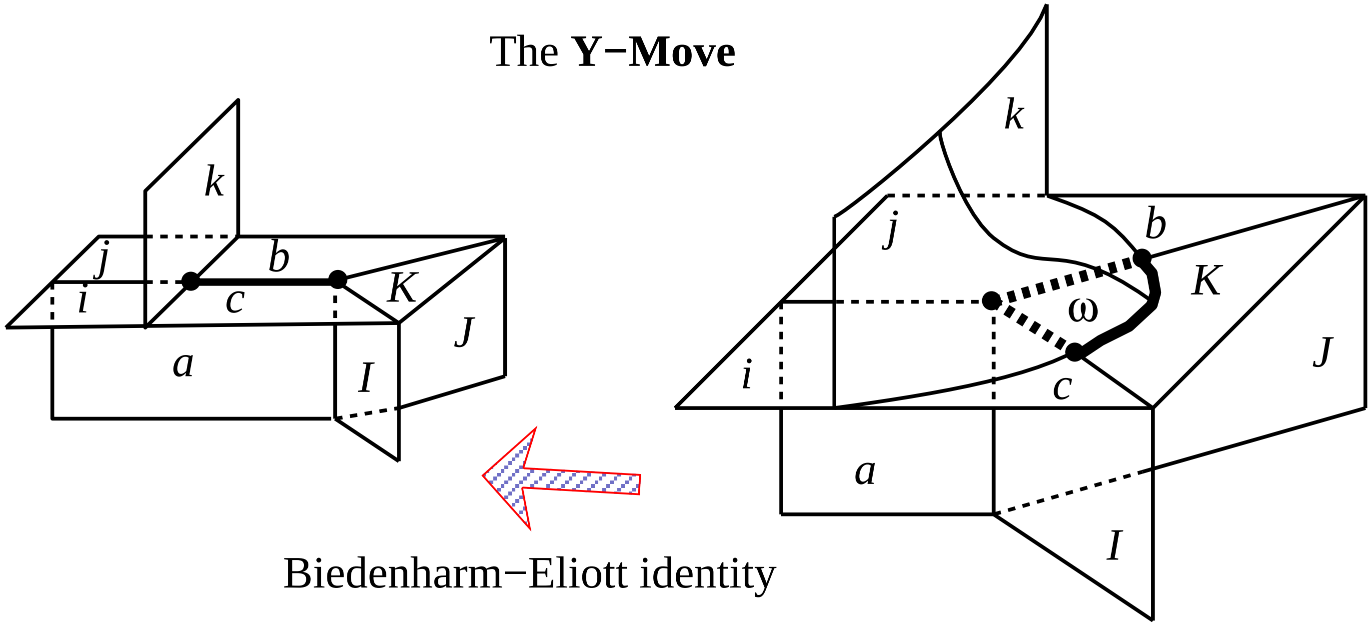


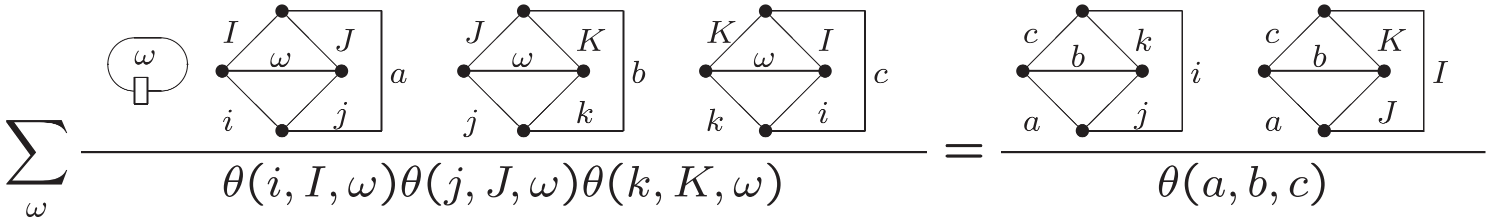
Δ	Δ^*		Factor in $TV_G[\mathcal{S}_2(\Delta^*)]$
vertex	3-cell	Type 0	$\Lambda^{-1} = \left[\sum_a \Delta_a\right]^{(-1)}$
edge	face	Type **I**	$\Delta_a = dim(a) =$ 
face	edge	Type **II**	$\theta(a,b,c)^{-1} = \Big[$  $\Big]^{(-1)}$
tetrahedron	vertex	Type **III**	$\mathrm{Tet}[abed; cf] =$ 

The Turaev-Viro model is easy

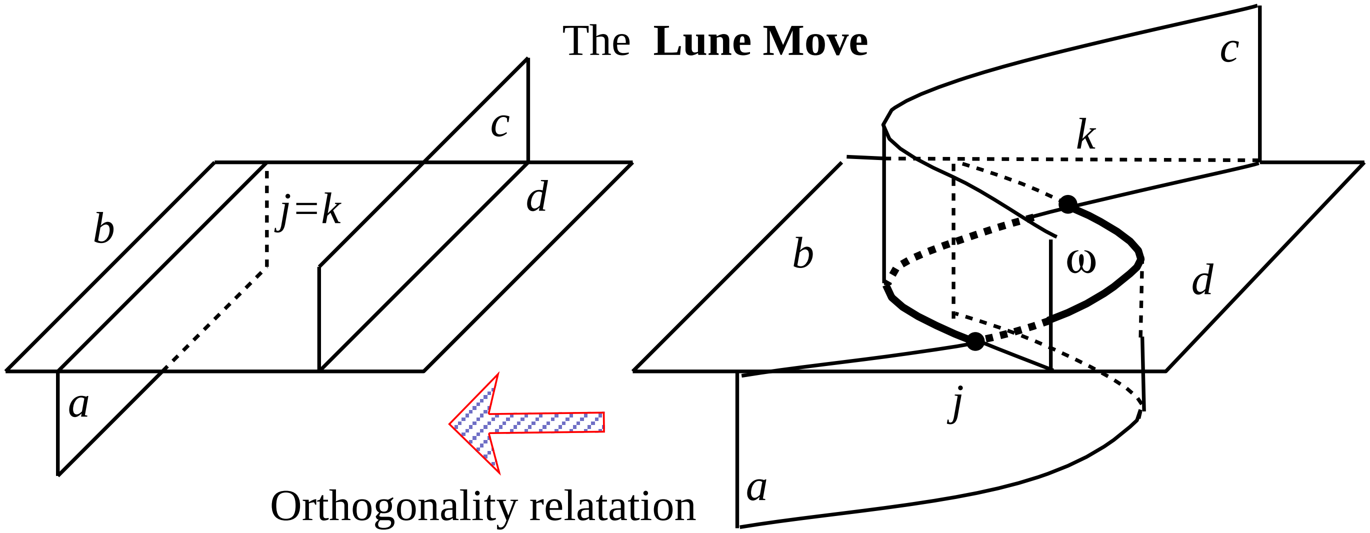
It is topological invariant !!

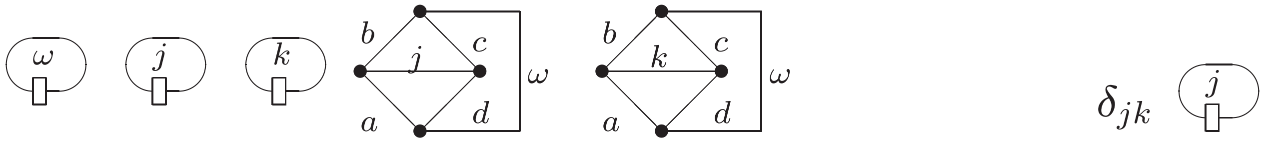
Invariance (1)





Invariance (2)

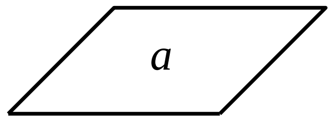


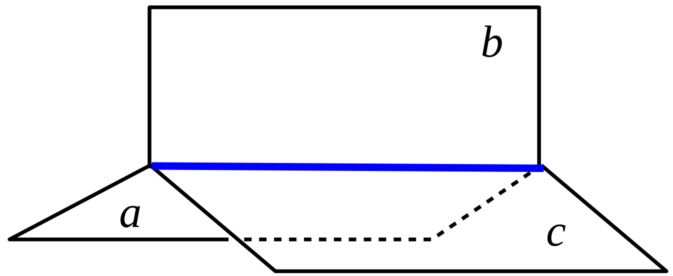


$$\sum_{\omega} \frac{\cdots}{\theta(a,b,j)\theta(b,c,\omega)\theta(a,b,k)\theta(c,d,j)\theta(a,d,\omega)\theta(c,d,k)} = \frac{\delta_{jk}\ \cdots}{\theta(a,b,j)\theta(c,d,j)} \tag{10}$$

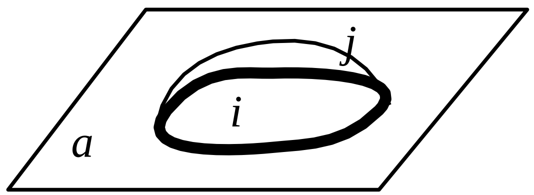
Invariance (3)

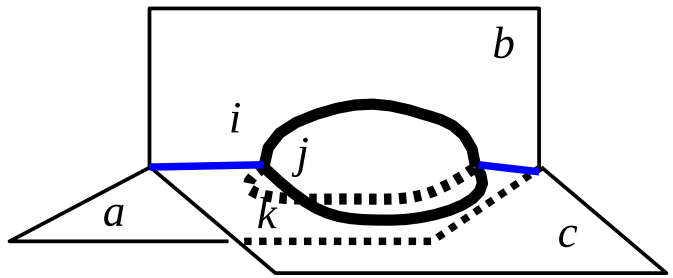
The following move are divergent!!!





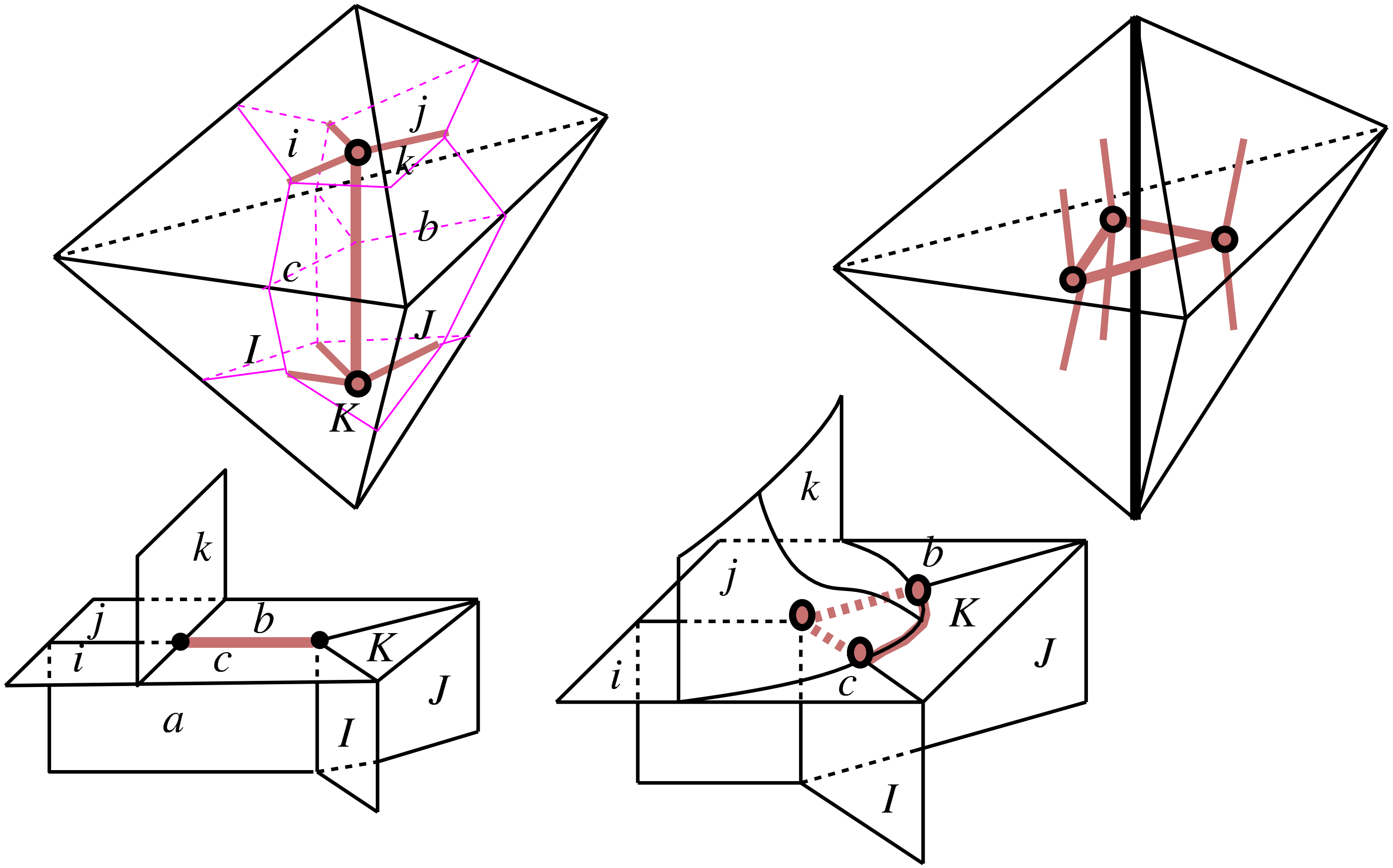
The **Bubble Move** **Edge Dilatation**



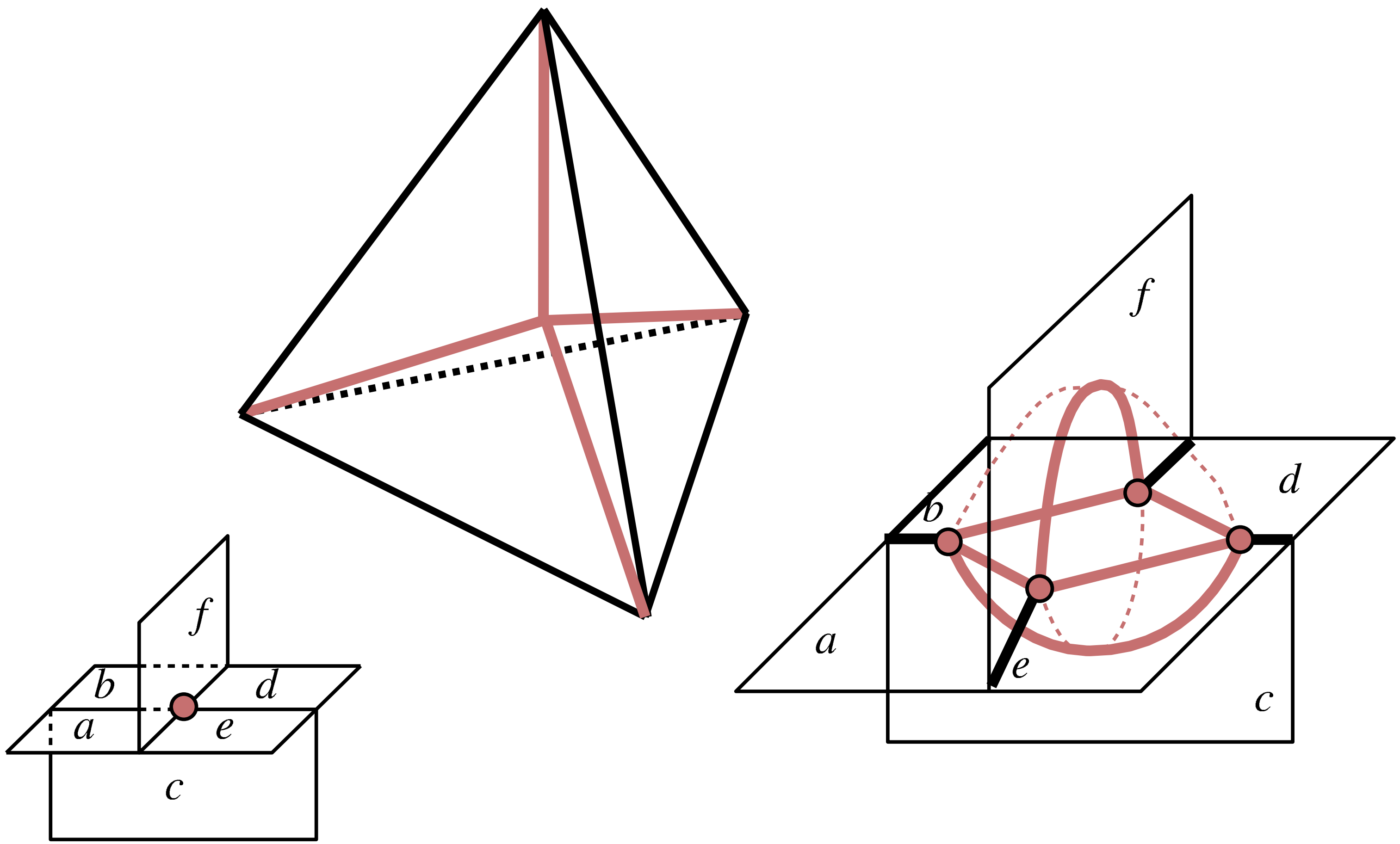


A regularization is needed. For example quantum groups !!

Pachener 2-3 move



Pachener 1-4 move



Following Turaev and Viro '92, we note that it is PRTV-model we can reinterpred as a spin foan model with spin network on the boundary.

Split the manifold M in three disjoint components glued together along the common boundary and rewrite the partition function associate to any given triangulation as:

$$TV_G[M^3, \Delta] = \sum_{\vec{c}' \in D(\gamma,G)} \sum_{\vec{c} \in D(\gamma,G)} TV_G[\mathcal{S}_2^{\Delta_1^*}; (\gamma', c')] \frac{1}{N(\gamma', c')^2} TV_G[\mathcal{S}_2^{\Delta_2^*}; (\gamma' \cup \gamma, c + c')] \frac{1}{N(\gamma, c)^2} TV_G[\mathcal{S}_2^{\Delta_3^*}; (\gamma, c)] \quad (11)$$

we are indeed inducing a coloring of γ from the coloring of the branched polyhedron. Moreover, if Δ is a triangulation of M then the resulting graph is trivalent.

Using completness of $\mathcal{H}_\gamma$ given by $\hat{1}_\gamma = \sum_{\vec{c} \in D(\gamma,G)} \frac{1}{N(\gamma,\vec{c})^2} |\Sigma; \gamma, \vec{c}\rangle \langle \Sigma; \gamma, \vec{c}|$ we can define an operator $\widehat{P}^{(M_2)}_{\gamma_2,\gamma_1}$ as:

$$\widehat{P}^{M_2}_{\gamma_2,\gamma_1} : \mathcal{H}_{\gamma_1} \Longrightarrow \mathcal{H}_{\gamma_2} \qquad \langle \gamma', c' | \widehat{P}^{M_2}_{\gamma',\gamma} | \gamma, c\rangle = TV_G[\mathcal{S}_2^{\Delta_2^*}; (\gamma' \cup \gamma, c + c')].$$

This operator $\widehat{P}^{M_2}_{\gamma',\gamma}$ does not depends on the interior of the simple polyhedron $\mathcal{S}_2^{\Delta_2^*}$ used to compute the transition amplitude [?]. Using this notation we can rewrite write equation (11) as

$TV_G[M^3,\Delta] = \langle\emptyset|\widehat{P}^{M_3}_{\emptyset,\gamma_2}\widehat{P}^{M_2}_{\gamma_2,\gamma_1}\widehat{P}^{M_1}_{\gamma_1,\emptyset}|\emptyset\rangle$. and we have: $\widehat{P}^{M_2\circ M_1}_{\gamma',\gamma} = \widehat{P}^{M_2}_{\gamma',\gamma''}\widehat{P}^{M_1}_{\gamma'',\gamma}$

This $\widehat{P}$ does not properly define a topological field theory since it is not a functor ($\widehat{P}^{\Sigma\times I}_{\gamma,\gamma} \neq 1_\gamma$). There is however a standard procedure that allows to define a functor from a semifunctor. It is sufficient to consider the Hilbert space $\overline{\mathcal{H}_\gamma} = \mathcal{H}_\gamma/Ker(\widehat{P}^{\Sigma\times I}_{\gamma,\gamma})$ from $\widehat{P}^{M}_{\gamma',\gamma} = \widehat{P}^{M}_{\gamma',\gamma}\circ\widehat{P}^{\Sigma\times I}_{\gamma,\gamma}$ follows that its restriction to the family of Hilbert spaces $\overline{\mathcal{H}_\gamma}$ become a functor. I.e, we used the operator $\widehat{P}^{\Sigma\times I}_{\gamma,\gamma}$ as the projector on the physical Hilbert space of the theory. In this way the PRTV-model defines a 2+1 dimensional quantum field theory in the Atiyah terminology.

From the physical point of view this discussion can be seen as a computation of Hartle-Hawking wave function and of the topology change amplitude of the 3-dimensional lattice gravity of Ponzano and Regge

SPIN FOAM MODELS: route (2) .....

The state sum formulation of the topological BF theory in 4-dimensions

$$
\begin{aligned}
Z_{BF}^{(4)} &= \int [\mathcal{D}A(x)][\mathcal{D}B(x)] e^{\mathrm{i}\int \mathrm{Tr}[B\wedge F[A]] + \Lambda\ \mathrm{Tr}[B\wedge B]} \\
&= \int [\mathcal{D}A(x)] e^{\frac{\mathrm{i}}{2\Lambda}\int \mathrm{Tr}[F[A]\wedge F[A]]} = \int [\mathcal{D}A(x)] e^{\frac{\mathrm{i}}{2\Lambda}\int d\left(\mathrm{Tr}[A\wedge F[A]] + \frac{2}{3}\mathrm{Tr}[A\wedge A\wedge A]\right)}
\end{aligned}
$$

with cosmological constant is given by the Crane-Yetter '97 or by the Ooguri '92 (without cosmological constant) models. The partition function of the model associated to a triangulation Δ of a manifold M can be written as :

$$
Z_{BF}(\Delta, \Lambda) = \sum_{j_f, i_t} \prod_f dim_q(j_f) \prod_v \phi_{q,v}(\vec{j}, \vec{\imath}),
$$

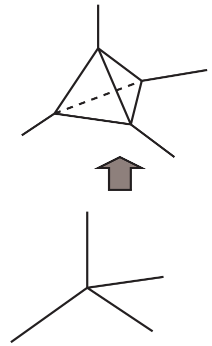
where $q = \exp(\mathrm{i}\Lambda)$, j_f denotes a coloring of the faces of Δ by irreducible representation of $U_q(su(2))$, i_t denotes a coloring of the tetrahedra of Δ by intertwiners and the sum is over all such colorings with $j < \frac{\pi}{\Lambda}$

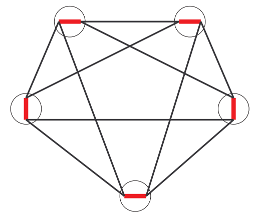
Barret-Crane model

The Relativistic-Spin-Foam model is a modification of the previous model for the $so(4)$ gauge group in which the conditions corresponding to the quantum 4-simplex conditions are imposed by hand.

$$Z_{BC}(\Delta,\Lambda) = \sum_{j_f,j'_f,i_t,i'_t} \prod_f dim_q(j_f) dim_{q^{-1}}(j'_f) \prod_v \phi_{q,v}(\vec{j},\vec{\imath}) \phi_{q^{-1},v}(\vec{j}',\vec{\imath}') \delta_{\vec{j},\vec{j}'} \delta_{\vec{\imath},\vec{\imath}'}.$$

This model corresponds to two copies of $su(2)$ BF theories together with 15 constraints imposed for each 4-simplex.
$\delta_{\vec{j},\vec{j}'}$ and $\delta_{\vec{\imath},\vec{\imath}'}$ corresponds to the geometricity constraint.





The associated 15J-symbol

De Pietri, Friedel, Reisemberger '98 have shown the correspond at the discretization of Plebanski action for GR

$$\mathcal{S}[\omega;B;\phi] = \int_M \left[B^{IJ} \wedge F_{IJ}(\omega) - \frac{\Lambda}{4} \epsilon_{IJKL} B^{IJ} \wedge B^{KL} - \frac{1}{2} \phi(B)_{IJ} \wedge B^{IJ} \right],$$

SPIN FOAM MODELS: route (3)

or .... Spin foam generating function

or .... Generalized matrix models

Motivations

Generalized matrix models and the associated PL-geometry of glued simplices are closely related to

- MATRIX MODEL ⇒ 2dim Quantum Gravity
- 3 TENSOR MODEL ⇒ Turaev-Viro Invariant of 3-Manifold
- 4 TENSOR MODEL ⇒ Quantum BF theory
- The Feynman graphs of the n-tensor model have weight proportional to the Dynamical Triangulation action.

Possible bridge between different approach to quantum gravity

General IDEA

Consider the partition function of Simplicial Quantum Gravity:

$$Z[G,\Lambda] = 1 + \sum_{\mathcal{T}\in \mathrm{Tria}(M)} w_n(\mathcal{T})\, \mathrm{e}^{-k_n \nu_n(\mathcal{T}) + h_{n-2}\nu_{n-2}(\mathcal{T})}$$

where the weight factor are given by:

$$k_n = \Lambda \cdot \mathrm{vol}(\sigma^n) + \frac{n(n+1)}{2} \frac{\arccos(1/n)}{16\pi G} \mathrm{vol}(\sigma^{n-2}), \qquad h_{n-2} = \frac{1}{8G} \mathrm{vol}(\sigma^{n-2}).$$

Question:

Is it possible to construct a generating functional for partition function?

$$Z[G,\Lambda] = \int [d\phi]\, \mathrm{e}^{-F[\phi,G,\Lambda]}$$

Matrix Model

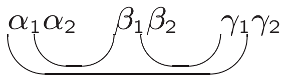
Partition Function: $Z[N,\lambda] = \int [d\phi]\ \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}[\phi^2] + \frac{\lambda}{3}\mathrm{Tr}[\phi^3]}$

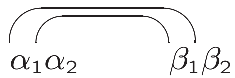
Measure $[d\phi] = \frac{1}{\mathcal{N}} \prod_{\alpha\leq\beta} d\Re\phi_{\alpha\beta} \prod_{\alpha<\beta} d\Im\phi_{\alpha\beta}$

$$Z[N,\lambda] = \exp\left[\frac{\lambda}{3} V^{\alpha_1\alpha_2;\beta_1\beta_2;\gamma_1\gamma_2} \frac{\delta}{\delta J^{\alpha_1\alpha_2}} \frac{\delta}{\delta J^{\beta_1\beta_2}} \frac{\delta}{\delta J^{\gamma_1\gamma_2}}\right] Z^{(0)}[N;J]\Big|_{J=0}$$

$$Z^{(0)}[N;J] = \int [d\phi]\ \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}[\phi^2] + \phi_{\alpha\beta}J^{\alpha\beta}} = \exp\left[\frac{1}{2} J^{\alpha\beta} G_{\alpha\beta;\gamma\delta} J^{\gamma\delta}\right],$$

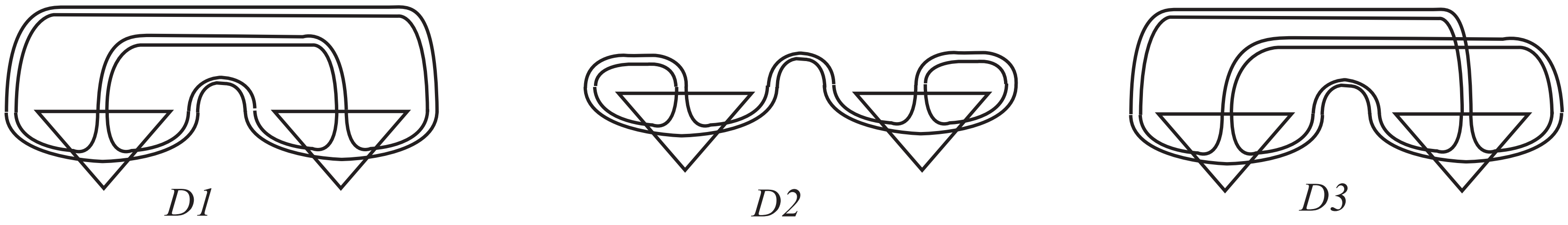
The propagator and the vertex are represented by the diagrams:

$V^{\alpha_1\alpha_2;\beta_1\beta_2;\gamma_1\gamma_2} \Rightarrow$ 

$G_{\alpha_1\alpha_2;\beta_1\beta_2} \Rightarrow$ 

The Feynman expantion, up to second order is:

$$
\begin{aligned}
Z[N,\lambda] &= 1+\lambda^2\left(\frac{1}{6}[[D1]]+\frac{1}{2}[[D2]]+\frac{1}{6}[[D3]]\right)+\dots\\
&= 1+\lambda^2\left(\frac{1}{6}N^3+\frac{1}{2}N^3+\frac{1}{6}N\right)+\dots\ .
\end{aligned}
$$



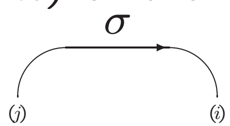
The n-tensor model partition function

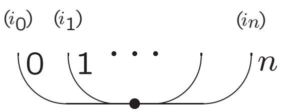
$$Z_n[N,\lambda] = \int [d\phi]\ \exp\left[-\frac{1}{2}\sum_{\alpha} |\phi_\alpha|^2 + \frac{\lambda}{n+1} \sum_{\alpha^{(0)}\ldots\alpha^{(n)}} V^{\alpha^{(0)}\ldots\alpha^{(n)}} \cdot \phi_{\alpha^{(0)}} \cdot \ldots \cdot \phi_{\alpha^{(n)}} \right] \qquad (12)$$

- Configuration Variables $\phi_{i_1\ldots i_n}$ with $\left(\phi_{\alpha_{\tau(1)}\ldots\alpha_{\tau(n)}} = \Re[\phi_{\alpha_1\ldots\alpha_n}] + i \cdot \mathrm{sgn}(\tau) \cdot \Im[\phi_{\alpha_1\ldots\alpha_n}]\right)$
- The Feynman diagram expantion is $\left(G_{\alpha_1\ldots\alpha_n;\beta_1\ldots\beta_n} = \frac{2}{n!} \sum_{\tau,\mathrm{sgn}(\tau)=-1} G^{(\tau)}_{\alpha_1\ldots\alpha_n;\beta_1\ldots\beta_n}, \right)$

$$Z_n[N,\lambda] = \sum_k \sum_{\sigma\in\mathfrak{S}_{\{0,\ldots,kn+k-1\}}} \frac{1}{k!}\frac{\lambda^k}{(n+1)^k}\frac{(1/2)^{k(n+1)/2}}{(k(n+1)/2)!} \times$$
$$\times V^{\alpha^{(0)}\ldots\alpha^{(n)}} \cdot \ldots \cdot V^{\alpha^{(kn+k-n-1)}\ldots\alpha^{(kn+k-1)}} \times G_{\alpha^{(\sigma(0))}\alpha^{(\sigma(1))}} \cdot \ldots \cdot G_{\alpha^{(\sigma(kn+k-2))}\alpha^{(\sigma(kn+k-1))}}$$

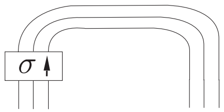
- Fat graphs $[\tau = \sigma \circ (1\ n)$ and σ odd.$]$

$G^{(\tau)}_{\alpha^{(j)}\alpha^{(i)}} \Rightarrow$ 

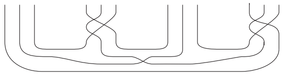
$V^{\alpha^{(i_0)}\ldots\alpha^{(i_n)}} \Rightarrow$ 

The three-tensor model

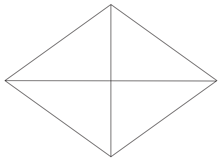
$$Z_3[N,\lambda] = \int [d\phi] \exp \left[-\frac{1}{2} \sum_{\alpha_1,\alpha_2,\alpha_3} |\phi_{\alpha_1\alpha_2\alpha_3}|^2 + \frac{\lambda}{4} \sum_{\alpha_1,\dots,\alpha_6} \phi_{\alpha_1\alpha_2\alpha_3}\phi_{\alpha_4\alpha_5\alpha_3}\phi_{\alpha_4\alpha_2\alpha_6}\phi_{\alpha_1\alpha_5\alpha_6} \right]$$



Propagator



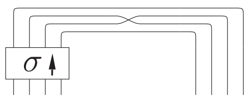
Vertex



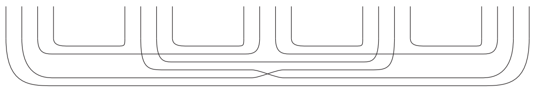
The tetrahedron

The four-tensor model

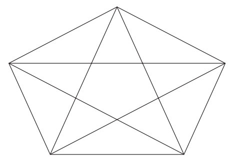
$$Z_4[N,\lambda] = \int [d\phi] \exp \Bigg[-\frac{1}{2} \sum_{\alpha_1,\dots,\alpha_4} |\phi_{\alpha_1\alpha_2\alpha_3\alpha_4}|^2 + \frac{\lambda}{5} \sum_{\alpha_1,\dots,\alpha_{10}} \phi_{\alpha_1\alpha_2\alpha_3\alpha_4}\phi_{\alpha_4\alpha_5\alpha_6\alpha_7}\phi_{\alpha_7\alpha_3\alpha_8\alpha_9}\phi_{\alpha_9\alpha_6\alpha_2\alpha_{10}}\phi_{\alpha_{10}\alpha_8\alpha_5\alpha_1} \Bigg]$$



Propagator



Vertex

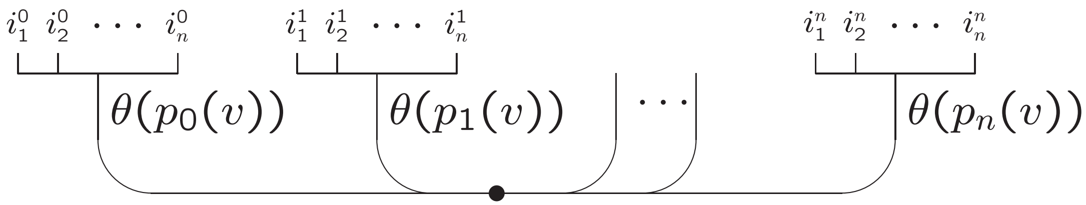


The 4-simplex

From Fat Graph to Glued Simplices

* to each vertex of G, an n-simplex $S(v)$ with labeled vertices $p_i(v)$ $(i = 0, \ldots, n)$.

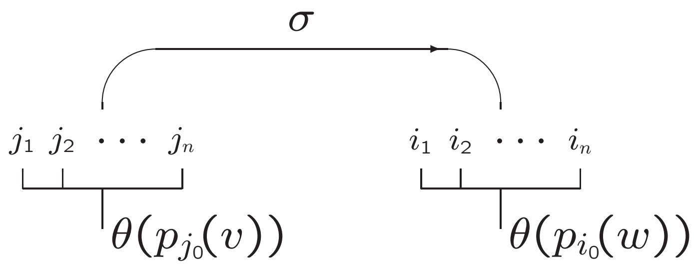
* a labeling of the corresponding codimension one faces:



(13)

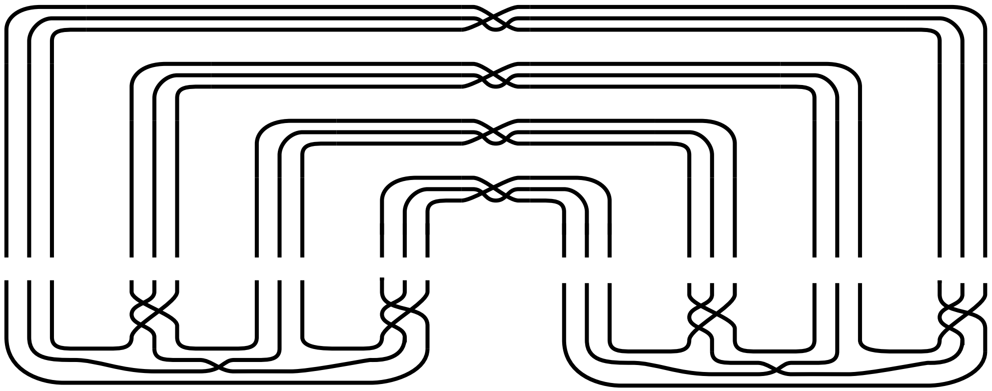
$(n \cdot k$ even $\Rightarrow (i_1^k, \cdots, i_n^k) = (k-1, \cdots, k-n)$, $n \cdot k$ odd $\Rightarrow (i_1^k, \cdots, i_n^k) = (k+1, \cdots, k+n)$, Indices meant modulo $n+1$.)

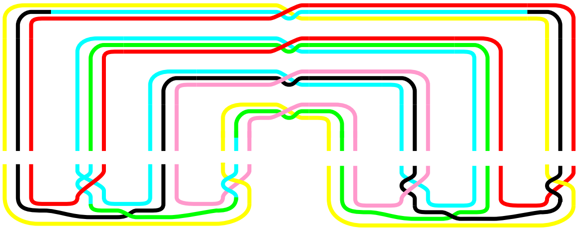
Now, each edge of G determines a pairing (simplicial identification) between the $(n-1)$-faces associated to its ends. In fact an edge of G defines a map from $\theta(p_{i_0}(v))$ to $\theta(p_{j_0}(w))$ which maps $p_{i_k}(v)$ to $p_{j_{\tau(k)}}(w)$ where $\tau = \sigma \circ (1\ n)$.



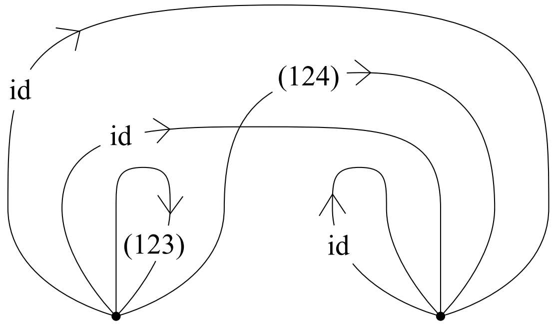
(14)

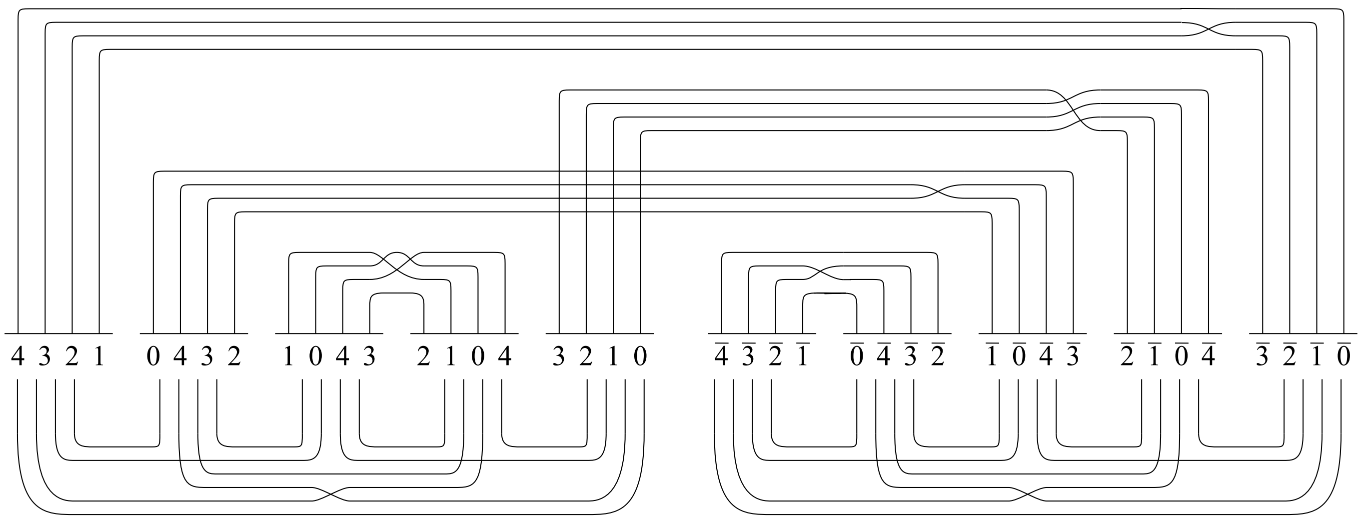
A three-dimensional example





A four-dimensional example





Feynman Graphs: $Z_n[N,\lambda] = 1 + \sum_{G \in \mathsf{GS}_n^+} w_n(G) \cdot \lambda^{\nu_0(G)} \cdot N^{\nu_2(G)}$.

where: $\nu_0(G)$ # of vertices of G, $\nu_2(G)$ # close circuits defined in G

When G defines a triangulation of a manifolds $\mathcal{T}$ $[h_{n-2} = \log N,\ k_n = -\log(\lambda)]$

$$Z_n[N,\lambda] = 1 + \sum_{M \in \mathsf{Top}_n^+} \sum_{\mathcal{T} \in \mathsf{Tria}(M)} w_n(\mathcal{T})\, \mathrm{e}^{-k_n \nu_n(\mathcal{T}) + h_{n-2}\nu_{n-2}(\mathcal{T})} + \sum_{G \in \mathsf{GS}_n^+,\ G \notin \mathsf{Top}_n} w_n(G) \cdot \lambda^{\nu_0(G)} \cdot N^{\nu_2(G)}.$$

The weight factor is the one of simplicial gravity

$$Z_n[N,\lambda] = 1 + \sum_{M \in \mathsf{Top}_n^+} \sum_{\mathcal{T} \in \mathsf{Tria}(M)} w_n(\mathcal{T})\, \mathrm{e}^{S(\Lambda,G,\mathcal{T})} + \dots$$

$$k_n = \Lambda \cdot \mathrm{vol}(\sigma^n) + \frac{n(n+1)}{2} \frac{\arccos(1/n)}{16\pi G} \mathrm{vol}(\sigma^{n-2}), \qquad h_{n-2} = \frac{1}{8G} \mathrm{vol}(\sigma^{n-2}).$$

The Boulatov Construction

- Replace the 3-tensor with a field

  $\phi_{i_1 i_2 i_3} \Rightarrow \phi(g_1, g_2, g_3)$

  1. $\phi : SU(2) \times SU(2) \times SU(2)/SU(2) \longrightarrow C$
  2. symmetry condition
  3. reality condition

- Mode expantions

$$\phi(g_1, g_2, g_3) \quad = \quad \sum_{J_1 J_2 J_3} \phi^{\alpha_1 \alpha_2 \alpha_3}_{J_1 J_2 J_3} \begin{pmatrix} J_1 & J_2 & J_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \times D^{\beta_1}_{(J_1)\alpha_1}(g_1) D^{\beta_2}_{(J_2)\alpha_2}(g_2) D^{\beta_3}_{(J_3)\alpha_3}(g_3)$$

- $Z[SU(2), \lambda] = 1 + \sum_G w_3(G) \; Z_{TV}[G]$

  Can be done using quantum groups

$$\int dg \phi(g_1 g, g_2 g, g_3 g) = \sum_{J_1 J_2 J_3} \phi^{\alpha_1\alpha_2\alpha_3}_{J_1 J_2 J_3\ \beta_1\beta_2\beta_3} \times D^{\gamma_1}_{(J_1)\alpha_1}(g_1) D^{\gamma_2}_{(J_2)\alpha_2}(g_2) D^{\gamma_3}_{(J_3)\alpha_3}(g_3)$$

$$\times D^{\beta_1}_{(J_1)\gamma_1}(g) D^{\beta_2}_{(J_2)\gamma_2}(g) D^{\beta_3}_{(J_3)\gamma_3}(g)$$

$$= \sum_{adm(J_1 J_2 J_3)} \phi^{\alpha_1\alpha_2\alpha_3}_{J_1 J_2 J_3} \begin{pmatrix} J_1 & J_2 & J_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \times D^{\beta_1}_{(J_1)\alpha_1}(g_1) D^{\beta_2}_{(J_2)\alpha_2}(g_2) D^{\beta_3}_{(J_3)\alpha_3}(g_3)$$

$$\int dg_1 dg_2 dg_3 \phi(g_1, g_2, g_3) \phi(g_1, g_2, g_3)$$

$$= \sum_{adm(J_1 J_2 J_3)} \sum_{\alpha_1\alpha_2\alpha_3} \sum_{\beta_1\beta_2\beta_3} \phi^{\alpha_1\alpha_2\alpha_3}_{J_1 J_2 J_3} \phi^{\beta_1\beta_2\beta_3}_{J_1 J_2 J_3}\ g^{(J_1)}_{\alpha_1\beta_1}\ g^{(J_2)}_{\alpha_2\beta_2}\ g^{(J_3)}_{\alpha_3\beta_3}$$

Going to dimension 4, Ooguri 1992

Do the same (like in dim=3)

Replace the 4-tensor with a Field $\phi_{i_1 i_2 i_3 i_4} \Rightarrow \phi(g_1, g_2, g_3, g_4)$

- $\phi : SU(2) \times SU(2) \times SU(2) \times SU(2)/SU(2) \longrightarrow C$
- symmetry condition
- reality condition

Mode expantions

$$\phi(g_1, g_2, g_3, g_4) = \sum_{J_1 J_2 J_3 J_4 M} \phi^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J_1 J_2 J_3 J_4 M} \begin{pmatrix} J_1 & J_2 & J_3 & J_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}_{[M]}$$

$$\times D^{\beta_1}_{(J_1)\alpha_1}(g_1) D^{\beta_2}_{(J_2)\alpha_2}(g_2) D^{\beta_3}_{(J_3)\alpha_3}(g_3) D^{\beta_4}_{(J_g)\alpha_4}(g_4)$$

- $Z[SU(2), \lambda] = 1 + \sum_G w_4(G)\ Z_{CYO}[G]$

(Application to the Barret-Crane Model)

Derive the spin foam model from a field theory over a group space.

A generic spin foam is identified as a given Feynman graph of the corresponding *group* field theory

The Barret-Craner model (for example) correspond to the Feynamn graph expantion of the following field theory:

$$S[\phi] = \frac{1}{2}\int\prod_{i=1}^{4}dx_i\ \phi^2(x_1,x_2,x_3,x_4) + \frac{\lambda}{5!}\int\prod_{i=1}^{10}dx_i\ \ \phi(x_1,x_2,x_3,x_4)$$
$$\phi(x_4,x_5,x_6,x_7)\ \phi(x_7,x_3,x_8,x_9)\ \phi(x_9,x_6,x_2,x_{10})\ \phi(x_{10},x_8,x_5,x_1).$$

The potential (fifth order) term has the structure of a 4-simplex where the following simmetry condition must be imposed $\phi(g_1,g_2,g_3,g_4) = \phi(g_1g,g_2g,g_3g,g_4g)$ ($\forall\ g \in$ SO(4)).

R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, Barrett-Crane model as an Boulatov-Ooguri theory over a homogeneous space. Nucl. Phys. B 574, (2000), 785-806. xxx-archive: hep-th/9907154.

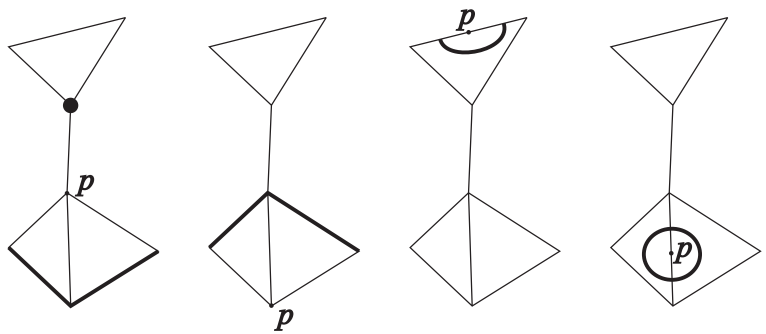
R. De Pietri, C. Petronio, Feynman diagrams of generalized matrix models and the associate manifolds in dimensions 4. Journal of Mathematical Physics 41, (2000), 6671-6688. xxx-archive: hep-th/0004045.

Question: Are the simplicial complexes obtained gluing together codimension 1-faces MANIFOLDS?

Which are the possible problems ?

Fact: A polihedron is a manifold if and only if the link of any point is an $(n-1)$-sphere

Example of the links of the various points p of a polihedron.



Observation 1: It is enough to check the baricenter of all the faces of the simplices.

Observation 2: There is nothing to check for codimension 1 faces.

Observation 3: We can avoid dealing with dimension 0 faces by removing the cones around the vertices.

Observation 4: In 3 dimenions we have to check only the links of the baricenters of the edges.

Observation 5: The gluing instruction translate into gluing instruction for the links.

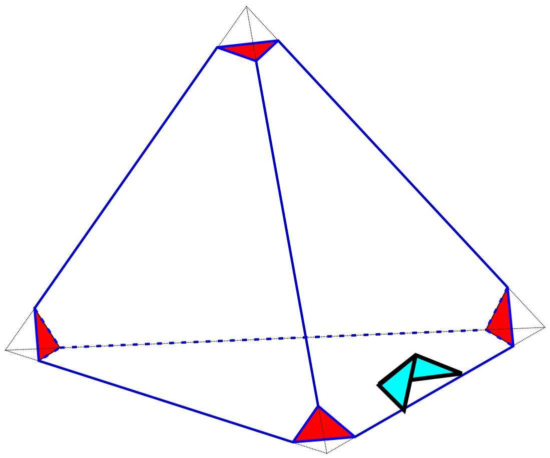
QUESTION: When do they are Manifolds ?

2dim ALWAYS

3dim Removing the cones around the vertices

$\Rightarrow$ They are ALWAYS compact manifold with boundary

The shaded areas, after gluing, will become the boundary components of X^{∂}

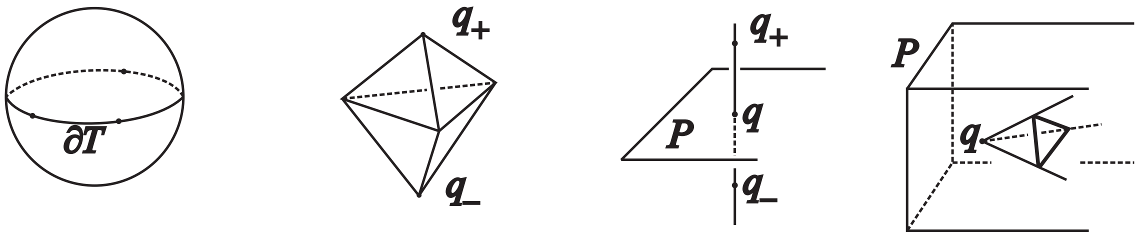


4dim 3 conditions on the Feynman graph MUST BE FULFILLED!!

They can be easily checked (algorithmically)

The 4 dimensional case

Links of the barycentres of a triangle (a) and an edge (b) in dimension four. The link of the midpoint of an edge is the double cone on the link in a cross-section (c).



(a) (b) (c)

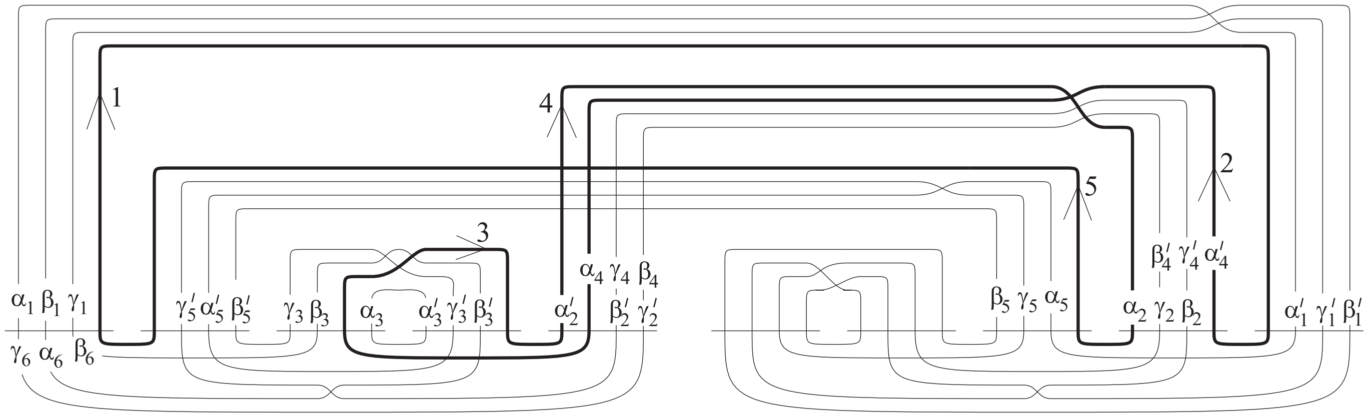
The triangle involved in the *Surf condition* is shaded.

Cycl Associated to each gluing there is permutation of three simbols $\{\sigma(0), \sigma(1), \sigma(2)\} = \{0, 1, 2\}$. .... At the end, we should have $\sigma = \text{id}$.

Surf To each gluing correspond a gluing of the triangles shown in (c). The closed surface $\Sigma(\mathcal{S}, \mathcal{P})$ resulting from these edge-pairings between the triangles should be an union of components homomorphic to S^2.

Dir To all the edges of the Δ_i's can be given an orientation in such a way that all the elements in $\mathcal{P}$, when restricted to edges, match the orientation.

The Cycl condition



We give arbitrary labels α, β, γ to the other three strands at the same junction, and we follow the labeling as we travel along the circuit, according to the following rules. First, when we travel through a propagator, we give matching strands the same label. Second, when we travel through a vertex, we note that the various strands come in groups of four, and we examine the relative position of the strand at which the circuit enters the vertex. If this position is the i-th one, then the circuit exits at the $(4-i)$-th position, and the labeling rules are as follows: $[\,|\,, \alpha, \beta, \gamma] \leftrightarrow [\alpha, \beta, \gamma, \,|\,]$ and $[\alpha, \beta, \,|\,, \gamma] \leftrightarrow [\gamma, \,|\,, \alpha, \beta]$, where the vertical segment $|$ represents the strand of circuit we are following. Now, the condition is that, when we come back to the starting junction, the labeling of the three other strands should be the same as at the beginning.

Conclusion

The geometry of 4 dimensional glued simplices can be controled

Matrix Model *like* formulation of quantum gravity

- Automatically include the sum over topologies
- Convergence of different approaches
- Space-Time as *condensation* of elementary building blocks
- Everything still Euclidean
- What to do with Feynman graphs that do not correspond to Manifolds
- How to compute $Z[N, \lambda]$?
- ....